# NEW TOOLS FOR INVESTIGATING POSITIVE MAPS IN MATRIX ALGEBRAS

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We provide a novel tool which may be used to construct new examples of positive maps in matrix algebras (or, equivalently, entanglement witnesses). It turns out that this can be used to prove positivity of several well-known maps (such as reduction map, generalized reduction, Robertson map, and many others). Furthermore, we use it to construct a new family of linear maps and prove that they are positive, indecomposable and (nd)optimal.

**Keywords:** entanglement witness, positive map.

## 1. Introduction

Quantum entanglement is one of the essential features of quantum physics and as a resource it is fundamental to modern applications of quantum mechanics like for example quantum teleportation and quantum cryptography [1, 2]. Therefore, there is a tremendous interest in developing efficient theoretical and experimental methods to detect entanglement. Linear positive maps in matrix algebras [3, 4] provide a basic tool to discriminate between separable and entangled states of composed quantum systems [2, 5]. A quantum state represented by the density operator $\rho$ living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if it can be represented as the following convex sum $\rho = \sum_\alpha p_\alpha \rho_\alpha^{(A)} \otimes \rho_\alpha^{(B)}$ where $p_\alpha$ denotes a probability distribution, and $\rho_\alpha^{(A)}$ and $\rho_\alpha^{(B)}$ are density operators of subsystems $A$ and $B$, respectively. It is well known [6] that $\rho$ represents a separable state if and only if $(\mathcal{I}_A \otimes \Lambda)\rho \geq 0$ for all linear positive maps $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$, where $\mathcal{I}_A : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ denotes an identity map, i.e., $\mathcal{I}_A(X) = X$ for each $X \in \mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H})$ denotes a $\mathbb{C}^*$-algebra of bounded operators in $\mathcal{H}$. Throughout the paper all Hilbert spaces are finite-dimensional and hence $\mathcal{B}(\mathcal{H})$ may be treated as a matrix algebra $\mathbb{M}_N(\mathbb{C}) \equiv \mathbb{M}_N$, where $\dim \mathcal{H} = N$.

Due to the well-known duality [7–9] between linear maps $\Phi : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ and linear operators in $\mathcal{H}_A \otimes \mathcal{H}_B$ one may equivalently formulate the separability problem in terms of entanglement witnesses [6, 10]. A Hermitian operator $\mathcal{W}$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called an *entanglement witness* if and only if: (1) $\mathrm{Tr}(\mathcal{W}\sigma_{\mathrm{sep}}) \geq 0$ for all separable states $\sigma_{\mathrm{sep}}$, and (2) there exists an entangled state $\rho$ such that $\mathrm{Tr}(\mathcal{W}\rho) < 0$ (one says that $\rho$ is detected by $\mathcal{W}$).

In what follows we concentrate on a class of so called indecomposable positive maps. Let us recall that a positive map $\Lambda$ is decomposable if $\Lambda = \Lambda_1 + \Lambda_2 \circ T$ where $\Lambda_1$ and $\Lambda_2$ are completely positive and $T$ denotes transposition in a given basis. Maps which are not decomposable are called indecomposable (or nondecomposable). Indecomposable maps play a prominent role in entanglement theory due to the fact that entangled positive partially transpose (PPT) states can be detected only *via* an indecomposable map, that is, if $\rho$ is PPT then $(\mathcal{I}_A \otimes \Lambda)\rho \geq 0$ for all decomposable maps $\Lambda$. Therefore, if for a PPT state $\rho$ one has $(\mathcal{I}_A \otimes \Lambda)\rho \ngeq 0$, then we are sure that $\rho$ is entangled and $\Lambda$ is indecomposable.

The central issue in this paper is the construction of optimal positive maps [11]. Recall that a positive map $\Lambda$ is *optimal* if and only if for any completely positive map $\Phi_{CP}$, the map $\Lambda - \Phi_{CP}$ is no longer positive. A positive map $\Lambda$ is called *nd-optimal* [11] if and only if for any decomposable map $\Phi_D$, the map $\Lambda - \Phi_D$ is no longer positive. Our knowledge of optimal positive maps is very limited. Recently this problem was investigated in [12–16]. It is clear that if $\Phi$ is nd-optimal then it is necessarily indecomposable. However, the converse is not true [15]. One may have an optimal indecomposable map which is not nd-optimal. Interestingly, to guarantee optimality it is sufficient to satisfy so-called *spanning property* [11]: an entanglement witness $W$ has a spanning property if a set of product vectors $\psi \otimes \phi \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\langle \psi \otimes \phi | \mathcal{W} | \psi \otimes \phi \rangle = 0,$$

spans the entire Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Now, if $W$ has a spanning property, then it is necessarily optimal. The converse is not true (the prominent example is the celebrated Choi map in $\mathbb{M}_3$ which is known to be extremal, and hence optimal, but does not have a spanning property [12, 16]). Let $\mathcal{W}^\Gamma$ be a partial transposition of $\mathcal{W}$. Clearly, if $\mathcal{W}$ is an entanglement witness so is $\mathcal{W}^\Gamma$. One has the following characterization of nd-optimality.

THEOREM 1.1 ([11]). *An entanglement witness $\mathcal{W}$ is nd-optimal if and only if both $\mathcal{W}$ and $\mathcal{W}^\Gamma$ are optimal.*

Optimal positive maps (or, equivalently, optimal entanglement witnesses) provide the most efficient tool to discriminate between separable and entangled states. It is well known that any entangled state may be detected by some optimal maps. In recent years there has been considerable effort in constructing and analyzing the structure of EWs. For some recent papers see e.g. [12–28]. In this paper we provide a novel tool which may be used to construct new examples of positive maps (entanglement witnesses). It is based on a class of positive matrices discussed in the next section. We show that it may be used to prove positivity of several well known maps (reduction map, generalized reduction, Robertson map and many others). Further, we provide a new family of maps and prove that they are positive, indecomposable, and even both optimal and nd-optimal.

The remainder of this paper is organized as follows. In Section 2 we present the proof of the main theorem. Section 3 contains an overview of applications of Theorem 2.1 to a few known positive maps. In Section 4 we present a new family of positive maps together with a prove of its important properties, such as indecomposability and (nd)optimality. Finally, we conclude in Section 5.

## 2. A class of positive semi-definite matrices

In this section we provide a class of positive definite matrices that enables one to construct positive maps in matrix algebras. Let us start by recalling a well-known lemma.

LEMMA 2.1 ([3, 4]). *A block matrix $M \in \mathbb{M}_{n+k}$,*

$$M = \begin{pmatrix} A & X \\ X^\dagger & B \end{pmatrix},$$

*with $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_k$ together with $A \geq 0$ and $B > 0$, is positive if and only if $A \geq XB^{-1}X^\dagger$.*

We shall use this result to prove the following.

THEOREM 2.1. *Let $\mathcal{M}_N^K$ be a matrix in $\mathbb{M}_{K\cdot N} = \mathbb{M}_N \otimes \mathbb{M}_K =: \mathbb{M}_N(\mathbb{M}_K)$ of the following form,*

$$\mathcal{M}_N^K = \left(\begin{array}{c|c|c|c} (1-\alpha_1)\,\mathbb{1}_K & -z_{12}M_{12} & \cdots & -z_{1N}M_{1N} \\ \hline -z_{12}^* M_{12}^\dagger & (1-\alpha_2)\,\mathbb{1}_K & \cdots & -z_{2N}M_{2N} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -z_{1N}^* M_{1N}^\dagger & -z_{2N}^* M_{2N}^\dagger & \cdots & (1-\alpha_N)\,\mathbb{1}_K \end{array}\right)$$

*with $\sum_{i=1}^N \alpha_i = 1$ $(0 \leq \alpha_i \leq 1$ for $i = 1, \ldots, N)$, $|z_{ij}| \leq 1$, and $M_{ij} \in \mathbb{M}_K(\mathbb{C})$, for $1 \leq i < j \leq N$ such that*

$$M_{ij}M_{ij}^\dagger = \alpha_j M_{ii}.$$

*If the blocks $M_{ij}$ of the matrix $\mathcal{M}_N^K$ satisfy the following properties:*

$$(1) \quad M_{ij}M_{kj}^\dagger = \alpha_j M_{ik},$$

$$(2) \quad M_{ii} \leq \alpha_i \,\mathbb{1}_K,$$

*then matrix $\mathcal{M}_N^K$ is positive semi-definite.*

*Proof*: We will perform a proof by induction with respect to the number of blocks $N$ in a matrix $\mathcal{M}_N^K$. Let us assume that the $\mathcal{M}_{N-1}^K$ matrix is positive. From Theorem 2.1 we know that to prove positivity of matrix $\mathcal{M}_N^K$ it is enough to show that the following inequality holds,

$$\mathcal{M}_{N-1}^K \geq \frac{\alpha_N}{1-\alpha_N}\mathcal{M}(\mathbf{z}, M_{ij}), \tag{1}$$

with

$$\mathcal{M}(\mathbf{z}, M_{ij}) := \left(\begin{array}{c|c|c|c} |z_{1N}|^2 M_{11} & z_{1N}z_{2N}^* M_{12} & \cdots & z_{1N}z_{N-1,N}^* M_{1,N-1} \\ \hline z_{2N}z_{1N}^* M_{12}^\dagger & |z_{2N}|^2 M_{22} & \cdots & z_{2N}z_{N-1,N}^* M_{2,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline z_{N-1,N}z_{1N}^* M_{1,N-1}^\dagger & z_{N-1,N}z_{2N}^* M_{2,N-1}^\dagger & \cdots & |z_{N-1,N}|^2 M_{N-1,N-1} \end{array}\right) \geq 0,$$

where the last inequality is a natural consequence of the construction. We introduce a normalization procedure for coefficients $\alpha_i$ in a following way,

$$\alpha_i' = \frac{\alpha_i}{1-\alpha_N}, \qquad \text{for } i = 1, \ldots, N-1,$$

where $\sum_{i=1}^{N-1} \alpha_i' = 1$. Applying this normalization for submatrices $M_{ij}$ gives us

$$M_{ij}' = \sqrt{\frac{\alpha_i'\alpha_j'}{\alpha_i\alpha_j}} M_{ij}, \qquad \text{for } 1 \leq i < j \leq N-1.$$

To show inequality (1) it is enough to prove that

$$\mathcal{M}_\beta \equiv \left(\begin{array}{c|c|c|c} B_1 & -z_{12}' M_{12}' & \cdots & -z_{1,N-1}' M_{1,N-1}' \\ \hline -z_{12}'^* (M_{12}')^\dagger & B_2 & \cdots & -z_{2,N-1}' M_{2,N-1}' \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -z_{1,N-1}'^* (M_{1,N-1}')^\dagger & -z_{2,N-1}'^* (M_{2,N-1}')^\dagger & \cdots & B_N \end{array}\right) \geq 0,$$

where

$$B_i = (1-\alpha_i'(1-\alpha_N))\mathbb{1}_K - |z_{iN}|^2 \alpha_N M_{ii}',$$
$$z_{ij}' = (1-\alpha_N) z_{ij} + \alpha_N z_{iN} z_{jN}^*,$$

with

$$|z_{ij}'| \leq (1-\alpha_N)|z_{ij}| + \alpha_N |z_{iN} z_{jN}^*| \leq 1.$$

A simple calculation shows that

$$M_{ij}'(M_{kj}')^\dagger = \frac{\alpha_j'}{\alpha_j}\sqrt{\frac{\alpha_i'\alpha_k'}{\alpha_i\alpha_k}}\, M_{ij} M_{kj}^\dagger = \frac{\alpha_j'}{\alpha_j}\sqrt{\frac{\alpha_i'\alpha_k'}{\alpha_i\alpha_k}}\, \alpha_j M_{ik} = \alpha_j'\, M_{ik}'.$$

By replacing $k$ with $i$ in the above formula one gets

$$M'_{ij}(M'_{ij})^\dagger = \alpha'_j\, M'_{ii}$$

and, due to the assumption,

$$M'_{ii} = \frac{\alpha'_i}{\alpha_i} M_{ii} \le \frac{\alpha'_i}{\alpha_i}\,\alpha_i\, \mathbb{1}_K = \alpha'_i\, \mathbb{1}_K.$$

The last inequality implies

$$B_i \ge (1 - \alpha'_i(1-\alpha_N))\,\mathbb{1}_K - |z_{iN}|^2 \alpha_N \alpha'_i\, \mathbb{1}_K \ge (1-\alpha'_i)\,\mathbb{1}_K.$$

As a consequence one finds

$$\mathcal{M}_\beta \ge \left(\begin{array}{c|c|c|c} (1-\alpha'_1)\,\mathbb{1}_K & -z'_{12}M'_{12} & \cdots & -z'_{1,N-1}M'_{1,N-1} \\ \hline -z'^*_{12}(M'_{12})^\dagger & (1-\alpha'_2)\,\mathbb{1}_K & \cdots & -z'_{2,N-1}M'_{2,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -z'^*_{1,N-1}(M'_{1,N-1})^\dagger & -z'^*_{2,N-1}(M'_{2,N-1})^\dagger & \cdots & (1-\alpha'_{N-1})\,\mathbb{1}_K \end{array}\right) = \mathcal{M}^K_{N-1}.$$

Since, by assumption, $\mathcal{M}^K_{N-1}$ is a positive matrix, we have completed the proof. □

## 3. New proofs of positivity for a series of linear maps

In this section we use Theorem 2.1 to provide new proofs of positivity for a series of well-known maps. Let us recall that to prove positivity of a given map $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ it is enough to show that each rank-1 projector $P \in \mathcal{B}(\mathcal{H}_A)$ is mapped *via* $\Lambda$ into a positive element in $\mathcal{B}(\mathcal{H}_B)$.

### 3.1. Generalized reduction map

Let us start our consideration with a generalized reduction map, $\mathcal{R}^{\mathbf{z}}_N : \mathbb{M}_N \to \mathbb{M}_N$, defined by

$$\mathcal{R}^{\mathbf{z}}_N(e_{ij}) = \frac{1}{N-1}\begin{cases} \mathbb{1}_N - e_{ii} & \text{for } i = j, \\ -z_{ij}e_{ij} & \text{for } i < j, \end{cases}$$

where $e_{ij} \in \mathbb{M}_N$ stands for fixed orthonormal basis and $\mathbf{z} = \{z_{12}, z_{13}, \ldots, z_{N-1,N}\}$ denotes a vector of complex numbers such that $|z_{ij}| \le 1$. Note that if $z_{ij} = 1$ then the above formula reproduces the standard normalized reduction map

$$\mathcal{R}_N(X) = \frac{1}{N-1}(\mathbb{1}_N \mathrm{Tr}(X) - X).$$

Let us consider a rank-1 projector $P_N = |\psi\rangle\langle\psi|$, with $\psi = \bigoplus_{i=1}^N \sqrt{\alpha_i}x_i$, and $x_i \in \mathbb{C}$, $\alpha_i \in [0,1]$, $\sum_{i=1}^N \alpha_i = 1$. Without loosing generality we can assume $|x_i|^2 = 1$ for all $i = 1, \ldots, N$. Now,

$$\mathcal{R}_N^{\mathbf{z}}(P_N) = \begin{bmatrix} 1-\alpha_1 & -z_{12}M_{12} & \cdots & -z_{1N}M_{1N} \\ -z_{12}^*M_{12}^\dagger & 1-\alpha_2 & \cdots & -z_{2N}M_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -z_{1N}^*M_{1N}^\dagger & -z_{2N}^*M_{2N}^\dagger & \cdots & 1-\alpha_N \end{bmatrix} = \mathcal{M}_N^1 \ ,$$

with

$$M_{ij} = \sqrt{\alpha_i\alpha_j}x_i x_j^*, \qquad \text{for } 1 \le i < j \le N.$$

As we can see $|z_{ij}| \le 1$ by definition of $\mathcal{R}_N^{\mathbf{z}}$ and for all off-diagonal blocks of the matrix $\mathcal{M}_N^1$ the following holds,

$$M_{ij}M_{kj}^\dagger = \left(\sqrt{\alpha_i\alpha_j}x_i x_j^*\right)\left(\sqrt{\alpha_k\alpha_j}x_k x_j^*\right)^\dagger = \alpha_j\sqrt{\alpha_i\alpha_k}\,|x_j|^2 x_i x_k^* = \alpha_j M_{ik}$$

for all $1 \le i < j \le N$, in particular $M_{ij}M_{ij}^\dagger = \alpha_j M_{ii}$, and

$$M_{ii} = \sqrt{\alpha_i\alpha_i}x_i x_i^* \le \alpha_i \mathbb{1}_1.$$

We have shown that conditions (1) and (2) of Theorem 2.1 are satisfied for a matrix $\mathcal{M}_N^1$ which proves positivity of a generalized reduction map. It is worth mentioning that it is well known that $\mathcal{R}_N$, i.e., standard normalized reduction map, is completely co-positive (i.e. $\mathcal{R}_2 \circ T$ is completely positive) and hence positive.

### 3.2. Robertson map

Let us now consider an action of a well-known Robertson map [29, 30]

$$\Psi_{\text{Rob}}(X) = \frac{1}{2}\left(\begin{array}{c|c} \mathbb{1}_2\text{Tr}X_{22} & -[X_{12}+\mathcal{R}_2(X_{21})] \\ \hline -[X_{21}+\mathcal{R}_2(X_{12})] & \mathbb{1}_2\text{Tr}X_{11} \end{array}\right), \tag{2}$$

where $X_{ij} \in \mathbb{M}_2$ and $\mathcal{R}_2$ stands for a reduction map in $\mathbb{M}_2$, on a rank-1 projector $P_4 = |\psi\rangle\langle\psi|$, with $\psi = \sqrt{\alpha_1}\psi_1 \oplus \sqrt{\alpha_2}\psi_2$ ($\psi_i \in \mathbb{C}^2$, $\alpha_i \in [0,1]$ for $i = 1,2$ and $\alpha_1+\alpha_2 = 1$). Actually, a map given by Eq. (2) is slightly different from the original one, but is unitary equivalent to the one proposed by Robertson (with $U = \imath\sigma_y$). Again without loosing generality we assume $\langle\psi_i|\psi_i\rangle = 1$, $i = 1,2$. Rewriting the reduction map as $\mathcal{R}_2(X) = \sigma_y X^T \sigma_y^\dagger$ allows us to represent $\Psi_{\text{Rob}}(P_4)$ as

$$\Psi_{\text{Rob}}(P_4) = \frac{1}{2}\left(\begin{array}{c|c} (1-\alpha_1)\mathbb{1}_2 & -M_{12} \\ \hline -M_{12}^\dagger & (1-\alpha_2)\mathbb{1}_2 \end{array}\right),$$

with the off-diagonal blocks of the form

$$M_{12} = \sqrt{\alpha_1\alpha_2}\left[|\psi_1\rangle\langle\psi_2| + \sigma_y|\psi_2^*\rangle\langle\psi_1^*|\sigma_y^\dagger\right]$$

and $z_{12} = 1$. We want to check whether conditions (1) and (2) of Theorem 2.1 are satisfied. Since for any antisymmetric and unitary matrix $U$ one has $\langle\psi|U\psi^*\rangle = 0$, thus, in particular, for $\sigma_y$ one has $\langle\psi_1|\sigma_y\psi_1^*\rangle = \langle\psi_2|\sigma_y\psi_2^*\rangle = 0$, and as a consequence

$$M_{12}M_{12}^\dagger = \alpha_1\alpha_2\left[|\psi_1\rangle\langle\psi_1| + \sigma_y|\psi_1^*\rangle\langle\psi_1^*|\sigma_y^\dagger\right] = \alpha_2 M_{11}.$$

Now, because $|\psi_1\rangle$ and $\sigma_y|\psi_1^*\rangle$ are two normalized orthonormal vectors, they define an orthonormal decomposition of an identity matrix and thus

$$M_{ii} = \alpha_1\left[|\psi_1\rangle\langle\psi_1| + \sigma_y|\psi_1^*\rangle\langle\psi_1^*|\sigma_y^\dagger\right] = \alpha_i\,\mathbb{1}_2, \tag{3}$$

which completes the proof. □

### 3.3. Generalization of the Robertson map

Let us recall a generalization of the Robertson map [18] to the $\mathbb{M}_{4N}$ algebra, given by

$$\Psi_{4N}(X) = \frac{1}{2N}\left(\begin{array}{c|c} \mathbb{1}_{2N}\mathrm{Tr}X_{22} & -[X_{12} + UX_{21}^T U^\dagger)] \\ \hline -[X_{21} + UX_{12}^T U^\dagger)] & \mathbb{1}_{2N}\mathrm{Tr}X_{11} \end{array}\right),$$

with $U \in \mathbb{M}_{2N}$ denoting an arbitrary antisymmetric and unitary matrix. Acting with a map $\Psi_{4N}$ on a projector $P_{4N} = |\psi\rangle\langle\psi|$, with $\psi = \sqrt{\alpha_1}\psi_1 \oplus \sqrt{\alpha_2}\psi_2$ ($\psi_i \in \mathbb{C}^{2N}$, $\alpha_1, \alpha_2 \in [0,1]$, $\alpha_1 + \alpha_2 = 1$ and $\langle\psi_i|\psi_i\rangle = 1$), leads to

$$\Psi_{4N}(P_{4N}) = \frac{1}{2N}\left(\begin{array}{c|c} (1-\alpha_1)\,\mathbb{1}_{2N} & -M_{12} \\ \hline -M_{12}^\dagger & (1-\alpha_2)\,\mathbb{1}_{2N} \end{array}\right),$$

with $M_{ij} = \sqrt{\alpha_i\alpha_j}[|\psi_i\rangle\langle\psi_j| + U(|\psi_j\rangle\langle\psi_i|)^T U^\dagger]$ and $z_{ij} = 1$. Direct calculation shows that

$$M_{12}M_{12}^\dagger = \alpha_1\alpha_2\left[|\psi_1\rangle\langle\psi_1| + U|\psi_1^*\rangle\langle\psi_1^*|U^\dagger\right] = \alpha_2 M_{11}$$

and

$$M_{ii} = \alpha_i\left[|\psi_i\rangle\langle\psi_i| + U\,(|\psi_i\rangle\langle\psi_i|)^T\,U^\dagger\right] \le \alpha_i\,\mathbb{1}_{2N}.$$

This is what we sought out to be proved (the last inequality is a consequence of a simple fact that $|\psi_i\rangle$ and $U|\psi_i\rangle$ are two orthonormal vectors and can be completed to a full orthonormal decompoition of the identity).

### 3.4. Complex extension of the Robertson map [19]

Both conditions from Theorem 2.1 are satisfied by the off-diagonal blocks of a matrix obtained from acting on a rank-1 projector $P_{2N}$ with a map $\Psi_{2N} : \mathbb{M}_{2N} \to \mathbb{M}_{2N}$ defined as

$$\Psi_{2N}(X) = \frac{1}{2(N-1)}\left[\begin{array}{c|c|c|c} A_1 & -z_{12}B_{12} & \cdots & -z_{1N}B_{1N} \\ \hline -z_{12}^*B_{21} & A_2 & \cdots & -z_{2N}B_{2N} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -z_{1N}^*B_{N1} & -z_{2N}^*B_{N2} & \cdots & A_N \end{array}\right],$$

with

$$A_i = \mathbb{I}_2\,(\mathrm{Tr}X - \mathrm{Tr}X_{ii})\,, \qquad \text{for } i = 1, \dots, N,$$
$$B_{ij} = X_{ij} + \mathcal{R}_2(X_{ji}), \qquad \text{for } 1 \le i < j \le N,$$

and $|z_{ij}| \le 1$ for $1 \le i < j \le 2N$, that is,

$$\Psi_{2N}(P_{2N}) = \frac{1}{2(N-1)} \left[\begin{array}{c|c|c|c} (1-\alpha_1)\,\mathbb{I}_2 & -z_{12}M_{12} & \cdots & -z_{1N}M_{1N} \\ \hline -z_{12}^* M_{12}^\dagger & (1-\alpha_2)\,\mathbb{I}_2 & \cdots & -z_{2N}M_{2N} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -z_{1N}^* M_{1N}^\dagger & -z_{2N}^* M_{2N}^\dagger & \cdots & (1-\alpha_N)\,\mathbb{I}_2 \end{array}\right],$$

where $P_{2N} = |\psi\rangle\langle\psi|$ ($\psi = \bigoplus_{i=1}^N \sqrt{\alpha_i}\psi_i$ with $\psi_i \in \mathbb{C}^2$, $\sum_{i=1}^N \alpha_i = 1$, $\alpha_i \in [0,1]$ and $\langle\psi_i|\psi_i\rangle = 1$ for $i = 1, \dots, N$) and the off-diagonal blocks are defined as

$$M_{ij} = \sqrt{\alpha_i\alpha_j}\left[|\psi_i\rangle\langle\psi_j| + \mathcal{R}_2\left(|\psi_j\rangle\langle\psi_i|\right)\right].$$

Indeed, simple calculation leads to

$$M_{ij}M_{kj}^\dagger = \alpha_j\sqrt{\alpha_i\alpha_k}\left[|\psi_i\rangle\langle\psi_k| + \sigma_y|\psi_i^*\rangle\langle\psi_k^*|\sigma_y^\dagger\right] = \alpha_j M_{ik}.$$

In particular, $M_{ij}M_{ij}^\dagger = \alpha_j M_{ii}$. Also, analogously to Eq. (3), $M_{ii} = \alpha_i\,\mathbb{I}_2$.

## 4. A new class of maps in $\mathbb{M}_N \otimes \mathbb{M}_{2K}$

In this section we provide a new class of positive maps in $\mathbb{M}_{N\cdot 2K}$. Any matrix in $\mathbb{M}_{N\cdot 2K}$ may be represented as a block $N \times N$ matrix in $\mathbb{M}_N(\mathbb{M}_{2K})$. Let us define a map $\Psi : \mathbb{M}_{N\cdot 2K} \to \mathbb{M}_{N\cdot 2K}$ in the following way

$$\Psi(X) = \frac{1}{2K(N-1)} \left[\begin{array}{c|c|c|c} A_1 & -z_{12}B_{12} & \cdots & -z_{1N}B_{1N} \\ \hline -z_{12}^* B_{21} & A_2 & \cdots & -z_{2N}B_{2N} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -z_{1N}^* B_{N1} & -z_{2N}^* B_{N2} & \cdots & A_N \end{array}\right],$$

with

$$A_i = \mathbb{I}_{2K}\,(\mathrm{Tr}X - \mathrm{Tr}X_{ii})\,, \qquad i = 1, \dots, N,$$
$$B_{ij} = X_{ij} + UX_{ji}^T U^\dagger, \qquad 1 \le i < j \le N,$$

with $U \in \mathbb{M}_{2K}$ denoting an arbitrary unitary and antisymmetric matrix and $|z_{ij}| \le 1$.

### 4.1. Positivity

PROPOSITION 4.1. *$\Psi$ defines a positive map.*

*Proof*: The first problem we want to tackle is positivity of the map $\Psi$. Let us consider a rank-1 projector $P = |\psi\rangle\langle\psi|$, where $\psi \in \mathbb{C}^{N\cdot 2K}$ denotes an arbitrary vector. Since $\mathbb{C}^{N\cdot 2K} = \bigoplus_{i=1}^{N} \mathbb{C}^{2K}$, we can represent $\psi$ as follows

$$|\psi\rangle = \bigoplus_{i=1}^{N} \sqrt{\alpha_i}|\psi_i\rangle,$$

with $\psi_i \in \mathbb{C}^{2K}$, $\alpha_i \in [0,1]$ for $i = 1, \ldots, N$ and $\sum_{i=1}^{N} \alpha_i = 1$. In addition, for simplicity, we can assume that $\langle\psi_i|\psi_i\rangle = 1$ for $i = 1, \ldots, N$, and then

$$\Psi(P) = \frac{1}{2K(N-1)} \left[ \begin{array}{c|c|c|c} (1-\alpha_1)\mathbb{1}_{2K} & -z_{12}M_{12} & \cdots & -z_{1N}M_{1N} \\ \hline -z_{12}^* M_{21} & (1-\alpha_2)\mathbb{1}_{2K} & \cdots & -z_{2N}M_{2N} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -z_{1N}^* M_{N1} & -z_{2N}^* M_{N2} & \cdots & (1-\alpha_N)\mathbb{1}_{2K} \end{array} \right],$$

with

$$M_{ij} = \sqrt{\alpha_i\alpha_j}\big[|\psi_i\rangle\langle\psi_j| + U|\psi_i^*\rangle\langle\psi_j^*|U^\dagger\big].$$

We want to check whether all conditions from Theorem 2.1 are satisfied. Taking into account that $U$ is a unitary and antisymmetric matrix one has $\langle\psi_i|U\psi_i^*\rangle = \langle\psi_i^* U^\dagger|\psi_i\rangle = 0$, and thus direct calculation leads to

$$M_{ij}M_{ik}^\dagger = \alpha_j\sqrt{\alpha_i\alpha_k}\big[|\psi_i\rangle\langle\psi_k| + U|\psi_i^*\rangle\langle\psi_k^*|U^\dagger\big] = \alpha_j M_{ik}.$$

By replacing $k$ with $j$ one gets $M_{ij}M_{ij}^\dagger = \alpha_j M_{ii}$. Moreover, since vectors $|\psi_i\rangle$ and $U|\psi_i^*\rangle$ are mutually orthogonal and normalized, one gets

$$M_{ii} = \alpha_i\big[|\psi_i\rangle\langle\psi_i| + U(|\psi_i\rangle\langle\psi_i|)^T U^\dagger\big] \le \alpha_i \mathbb{1}_{2K}.$$

Therefore we have proved that $\Psi(P)$ is positive semi-definite, which completes the proof. □

### 4.2. Indecomposibility

In order to prove that a given map is indecomposable it is enough to find an entangled PPT (i.e. positive partial transpose) state $\rho$ such that $\mathrm{Tr}(\mathcal{W}\rho) < 0$. Let $\mathcal{W}_\Psi$ be an EW corresponding to a positive map $\Psi$,

$$\mathcal{W}_\Psi = \frac{1}{d}\sum_{i,j=1}^{d} e_{ij} \otimes W_{ij},$$

where $W_{ij} = \Psi(e_{ij})$ and, to simplify notation, we denote $d := N \cdot 2K$. Let us consider a following construction for the state $\rho$,

$$\rho = \frac{1}{2k+1} \sum_{i,j=1}^{d} e_{ij} \otimes \rho_{ij}, \tag{4}$$

where the diagonal blocks of $\rho$ are given by

$$\rho_{ii} = \frac{\mathbb{1}_d}{d} - \big(2K(N-1)-1\big) W_{ii} \qquad \text{for } i = 1, \ldots, d,$$

and for the off-diagonal blocks one has

(1) if $0 < |i-j| < 2K$, then $\rho_{ij} = \mathbb{O}_d$,

(2) if $|i-j| = 2K\ell$, where $\ell = 1, \ldots, (N-1)$, then for each $\ell$ and $i = 1, \ldots, 2K(N-\ell)$ one has

$$\rho_{i,i+2K\ell} = -W_{i,i+2K\cdot\ell} \ ,$$

(3) if $|i-j| > 2K$, with $|i-j| \neq 2K\ell$, then

$$\rho_{ij} = \widetilde{e}_{ij} = \frac{z_{ij}}{(2K)^2 N(N-1)} e_{ij} \ ,$$

where $\{e_{ij}\}_{i,j=1}^{d}$ stands for an orthonormal basis in $\mathbb{M}_d$.

PROPOSITION 4.2. *A matrix $\rho$ defined by the above construction represents a PPT state, i.e. $\rho \geq 0$ and $\rho^{\Gamma} \geq 0$.*

PROPOSITION 4.3. *If $|z_{ij}| = 1$, then a map $\Psi$ is indecomposible.*

*Proof*: We will show that $\mathrm{Tr}(\mathcal{W}_\Psi \rho) = \sum_{i,j=1}^{d} \mathrm{Tr}(W_{ij}\rho_{ji}) < 0$. One has

$$\mathrm{Tr}(\mathcal{W}_\Psi \rho) = \frac{1}{\mathcal{N}} \Big\{ \sum_{|i-j|=2K\cdot\ell} \mathrm{Tr}(W_{ij}\rho_{ji}) + \sum_{0<|i-j|<2K} \mathrm{Tr}(W_{ij}\rho_{ji}) + \sum_{|i-j|>2K} \mathrm{Tr}(W_{ij}\rho_{ji}) \Big\}.$$

The first sum consists of $N$ terms, the second sum has $2K-1$ terms, and the last one – $(2K-1)(N-1)$ terms. Straightforward algebra leads to

$$\sum_{|i-j|=2K\cdot\ell} \mathrm{Tr}\left(W_{ij}\rho_{ji}\right) = \frac{2(K-1)}{(2K)^3 N(N-1)} \ ,$$

$$\sum_{0<|i-j|<2K} \mathrm{Tr}\left(W_{ij}\rho_{ji}\right) = 0,$$

$$\sum_{|i-j|>2K} \mathrm{Tr}\left(W_{ij}\rho_{ji}\right) = -\frac{(2K-1)}{(2K)^3 N(N-1)} \ ,$$

and, as a consequence,

$$\mathrm{Tr}\left(\mathcal{W}_\Psi \rho\right) = -\frac{1}{(2K+1)(2K)^3 N(N-1)} < 0,$$

which completes the proof. □

### 4.3. Optimality

In previous section we have shown that if $|z_{ij}| = 1$ for all $1 \le i < j < d$, then the map $\Psi$ is non-decomposable. It turns out that the same condition is necessary and sufficient for optimality.

PROPOSITION 4.4. *$\Psi$ is an optimal map if and only if $|z_{ij}| = 1$ for all $1 \le i < j \le N$.*

*Proof*: To show that $|z_{mn}| = 1$ is a necessary condition for optimality we may apply the same argument as in [19]. To show that $|z_{mn}| = 1$, that is, $z_{mn} = \mathrm{e}^{\imath\alpha_{mn}}$ [we introduced $\imath$ as an imaginary unit], is a sufficient condition it is enough to consider a set of vectors $\Gamma_\Psi$ defined as

$$\Gamma_\Psi = \{e_k \otimes e_k,\ \varphi_{mn} \otimes \varphi_{mn},\ \phi_{mn} \otimes \widetilde{\phi}_{mn};\ \text{ for } k = 1, \dots, d \text{ and } 1 \le m < n \le d\}, \quad (5)$$

where

$$\varphi_{mn} := e_m + \mathrm{e}^{-\imath\frac{\alpha_{mn}}{2}} e_n, \qquad \phi_{mn} := e_m + \imath\mathrm{e}^{-\imath\frac{\alpha_{mn}}{2}} e_n, \qquad \widetilde{\phi}_{mn} := e_m - \imath\mathrm{e}^{-\imath\frac{\alpha_{mn}}{2}} e_n$$

From simple linear algebra it follows that elements of the set $\Gamma_\Psi$ are linearly independent. Direct calculation shows that for each $\psi_l \otimes \widetilde{\psi}_l \in \Gamma_\Psi$ the following holds,

$$\langle \psi_l \otimes \widetilde{\psi}_l | \mathcal{W}_\Psi | \psi_l \otimes \widetilde{\psi}_l \rangle = 0 \ ,$$

which proves the theorem. □

### 4.4. Nd-optimality

PROPOSITION 4.5. *$\Psi$ defines a family of nd-optimal maps.*

*Proof*: According to Theorem 1.1 to show that $\mathcal{W}$ is nd-optimal it is enough to show, that both $\mathcal{W}$ and $\mathcal{W}^\Gamma$ are optimal entanglement witnesses. In Proposition 4.4 we have proved that $\mathcal{W}_\Psi$ is optimal. Now we will show that $\mathcal{W}_\Psi^\Gamma$ is an optimal EW as well. Let us consider the following transformation:

$$(\mathbb{1}_d \otimes V)\mathcal{W}_\Psi(\mathbb{1}_d \otimes V^\dagger) = \sum_{i,j=1}^{d} e_{ij} \otimes V\Psi(e_{ij})V^\dagger,$$

where $V := \mathbb{1}_N \otimes U^\dagger$. The action of $\Psi$ on basis elements $\{e_{ij}\}_{i,j=1}^d$ is given by

$$\Psi(e_{ij}) = \frac{1}{2K \cdot (N-1)} \begin{cases} (\mathbb{1}_N - e_{pp}^{(N)}) \otimes \mathbb{1}_{2K}\mathrm{Tr}(e_{rs}^{(2K)}), & \text{if } q = p, \\ -z_{ij}\big[e_{pq}^{(N)} \otimes e_{rs}^{(2K)} + e_{qp}^{(N)} \otimes U(e_{rs}^{(2K)})^T U^\dagger\big], & \text{if } q \ne p, \end{cases}$$

where we introduced vectors $e_i, e_j \in \mathbb{C}^d = \mathbb{C}^N \otimes \mathbb{C}^{2K}$ *via* the rule

$$\begin{cases} e_i = e_p^{(N)} \otimes e_r^{(2K)}, \\ e_j = e_q^{(N)} \otimes e_s^{(2K)}, \end{cases}$$

that is, each $i \in \{1, \ldots, d = N \cdot 2K\}$ defines a pair $(p, r)$ with $p \in \{1, \ldots, N\}$ and $r \in \{1, \ldots, 2K\}$. It is easy to check, that

$$V\Psi(e_{ij})V^\dagger = \frac{1}{2K \cdot (N-1)}(\mathbb{1}_N - e^{(N)}_{pp}) \otimes \mathbb{1}_{2K} \mathrm{Tr}(e^{(2K)}_{rs}),$$

for $q = p$ and

$$V\Psi(e_{ij})V^\dagger = \frac{-z_{ij}}{2K \cdot (N-1)}\big[e^{(N)}_{qp} \otimes (U^\dagger e^{(2K)}_{rs} U)^T + e^{(N)}_{pq} \otimes e^{(2K)}_{rs}\big]^T$$

otherwise. Since $U$ is a unitary and antisymmetric matrix, one has

$$(U^\dagger e_{rs} U)^T = U^T e^T_{rs} U^* = -U e^T_{rs} U^* = U e^T_{rs} U^\dagger$$

and, as a consequence, one gets $V\Psi(e_{ij})V^\dagger = \big(\Psi(e_{ij})\big)^T$. This shows that

$$(\mathbb{1}_d \otimes V)\mathcal{W}(\mathbb{1}_d \otimes V^\dagger) = \mathcal{W}^\Gamma.$$

Since for all $\psi_l \otimes \widetilde{\psi}_l \in \Gamma_\Psi$, where $\Gamma_\Psi$ is defined as (5), one has

$$\langle \psi_l \otimes V^\dagger \widetilde{\psi}_l | \mathcal{W}^\Gamma_\Psi | V\psi_l \otimes \widetilde{\psi}_l \rangle = \langle \psi_l \otimes \widetilde{\psi}_l | \mathcal{W}_\Psi | \psi_l \otimes \widetilde{\psi}_l \rangle = 0.$$

The independence of vectors $\{V\psi_l \otimes \widetilde{\psi}_l;\ l = 1, \ldots, d^2\}$ follows from the fact that $V$ is a unitary matrix. Indeed, one can rewrite the action of $V$ as

$$V\psi_l \otimes \widetilde{\psi}_l = (V \otimes \mathbb{1}_d)(\psi_l \otimes \widetilde{\psi}_l)$$

where $V \otimes \mathbb{1}_d$ is still a unitary matrix. As we know, vectors $\psi_l \otimes \widetilde{\psi}_l$ are linearly independent and thus one gets

$$\det\big[(V \otimes \mathbb{1}_d)([\psi_l \otimes \widetilde{\psi}_l]^{d^2}_{l=1})\big] = \det(V \otimes \mathbb{1}_d) \cdot \det([\psi_l \otimes \widetilde{\psi}_l]^{d^2}_{l=1}) \neq 0.$$

This completes the proof. □

## 5. Conclusions

We provided a new tool which may be used to construct new examples of positive maps (entanglement witnesses) in finite-dimensional matrix algebras. Interestingly, it allows to present a universal proof of positivity of several well-known maps (reduction map, generalized reduction, Robertson map and many others). Finally, it is shown that our method enables one to construct a new family of linear maps and prove that they are positive, indecomposable and even optimal. It should be stressed that this construction provides linear maps $\Psi : \mathbb{M}_d \to \mathbb{M}_d$ only for $d = 2N$. It would be interesting to find an analogous method if $d$ is odd. For example it would be desirable to provide an appropriate construction generalizing the well-known Choi map $\Psi_{\mathrm{Choi}} : \mathbb{M}_3 \to \mathbb{M}_3$ which was proved to be indecomposable and extremal. In a forthcoming paper we plan to report recent progress in this direction.

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