Constructing optimal entanglement witnesses

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We provide a class of indecomposable entanglement witnesses. In 4×4 case, it reproduces the well-known Breuer-Hall witness. We prove that these witnesses are optimal and atomic, i.e., they are able to detect the "weakest" quantum entanglement encoded into states with positive partial transposition. Equivalently, we provide a construction of indecomposable atomic maps in the algebra of $2k\times 2k$ complex matrices. It is shown that their structural physical approximations give rise to entanglement breaking channels. This result supports recent conjecture by Korbicz *et al.* [Phys. Rev. A **78**, 062105 (2008)].

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I. INTRODUCTION

The interest on quantum entanglement has dramatically increased during the last 2 decades due to the emerging field of quantum information theory [1]. It turns out that quantum entanglement may be used as basic resources in quantum information processing and communication. The prominent examples are quantum cryptography, quantum teleportation, quantum error correction codes, and quantum computation.

Since the quantum entanglement is the basic resource for the new quantum information technologies, it is therefore clear that there is a considerable interest in efficient theoretical and experimental methods of entanglement detection (see [2] for the review). The most general approach to characterize quantum entanglement uses a notion of an entanglement witness (*EW*) [3,4]. A Hermitian operator W defined on a tensor product $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$ is called an *EW* if and only if (1) $\mathrm{Tr}(W\sigma_{\mathrm{sep}})\geq 0$ for all separable states σ_{sep} and (2) there exists an entangled state ρ such that $\mathrm{Tr}(W\rho)<0$ (one says that ρ is detected by W). It turns out that a state is entangled if and only if it is detected by some *EW* [3]. There was a considerable effort in constructing and analyzing the structure of *EW*s [5–15]. In fact, entanglement witnesses have been measured in several experiments [16,17]. Moreover, several procedures for optimizing *EW*s for arbitrary states were proposed [6,18–20]. It should be stressed that there is no universal W, i.e., there is no entanglement witness which detects all entangled states. Each entangled state ρ may be detected by a specific choice of W. It is clear that each *EW* provides a new separability test and it may be interpreted as a new type of Bell inequality . There is, however, no general procedure for constructing *EW*s.

Due to the Choi-Jamiołkowski isomorphism [21,22] (actually, this isomorphism was analyzed already by de Pillis [23]), any *EW* corresponds to a linear positive map $\Lambda:\mathcal{B}(\mathcal{H}_A)\rightarrow\mathcal{B}(\mathcal{H}_B)$, where by $\mathcal{B}(\mathcal{H})$ we denote the space of linear operators on the Hilbert space $\mathcal{H}$ [throughout this paper, all Hilbert spaces are finite dimensional and therefore $\mathcal{B}(\mathcal{H})$ contains all linear operators from $\mathcal{H}$ to $\mathcal{H}$]. Recall that a linear map Λ is said to be positive if it sends a positive operator on $\mathcal{H}_A$ into a positive operator on $\mathcal{H}_B$. It turns out [3] that a state ρ in $\mathcal{H}_A\otimes\mathcal{H}_B$ is separable if and only if $(\mathbb{1}_A\otimes\Lambda)\rho$ is positive definite for all positive maps $\Lambda:\mathcal{B}(\mathcal{H}_B)\rightarrow\mathcal{B}(\mathcal{H}_A)$ (actually this result is based on [24]). Unfortunately, in spite of the considerable effort, the structure of positive maps is rather poorly understood [24–42].

In the present paper, we provide a construction of a class of positive maps in $\mathcal{B}(\mathbb{C}^{2k})$ with $k\geq 2$. Our construction uses the well-known reduction map as a building block. It turns out that for $k=2$, our construction reproduces Breuer-Hall maps [36,37] but for $k>2$, it gives completely new family of maps. It is shown that proposed maps are indecomposable [i.e., they are able to detect entangled positive partial transposition (PPT) states] and atomic (i.e., they are able to detect "weakly" entangled PPT states). As a by-product, we construct new families of PPT entangled states detected by our maps.

The paper is organized as follows. For pedagogical reason, we collect basic definitions and present the most important properties of positive maps and entanglement witnesses in Sec. II. Section III provides basic construction. Then in Sec. IV, we study basic properties of our maps and witnesses (indecomposability, atomicity, optimality). Section V discusses structural physical approximation (SPA) [43–45] of our maps. It is shown that the corresponding SPA gives rise to entanglement breaking channels and hence it supports recent conjecture by Korbicz *et al.* [45]. Final conclusions are collected in the last Sec. IV.

II. POSITIVE MAPS, ENTANGLEMENT WITNESSES, AND ALL THAT

For the reader's convenience, we recall basic definitions and properties which are important throughout this paper.

A. Positive maps

Let $\Lambda:\mathcal{B}(\mathcal{H}_A)\rightarrow\mathcal{B}(\mathcal{H}_B)$ be a positive linear map. In what follows, we shall consider only finite dimensional Hilbert spaces such that $\dim\mathcal{H}_A=d_A$ and $\dim\mathcal{H}_B=d_B$. One calls Λ k positive if

$$\mathbb{1}_k\otimes\Lambda:M_k\otimes\mathcal{B}(\mathcal{H}_A)\rightarrow M_k\otimes\mathcal{B}(\mathcal{H}_B) \qquad (1)$$

is positive. In the above formula, M_k denotes a linear space of $k\times k$ complex matrices and $\mathbb{1}_k:M_k\rightarrow M_k$ is an identity map, i.e., $\mathbb{1}_k(A)=A$ for each $A\in M_k$. A positive map which is k positive for each k is called completely positive (CP). Actually, if $d_A, d_B < \infty$, one shows [21] that Λ is CP if and only if it is d positive with $d = \min\{d_A, d_B\}$.

Definition 1 [25]. A positive map Λ is decomposable if

$$\Lambda = \Lambda_1 + \Lambda_2 \circ T, \tag{2}$$

where Λ_1 and Λ_2 are CP and T denotes transposition in a given basis. Maps which are not decomposable are called indecomposable (or nondecomposable).

Definition 2 [30]. A positive map Λ is atomic if it cannot be represented as

$$\Lambda = \Lambda_1 + \Lambda_2 \circ T, \tag{3}$$

where Λ_1 and Λ_2 are two-positive.

Definition 3 [6]. A positive map Λ is optimal if and only if for any CP map Φ, the map $\Lambda - \Phi$ is no longer positive.

B. Entanglement witnesses

Using Choi-Jamiołkowski isomorphism [21,22], each positive map Λ gives rise to entanglement witness W,

$$W = d_A(\mathbb{I}_A \otimes \Lambda)P_A^+, \tag{4}$$

where P_A^+ denotes maximally entangled state in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_A}$ and $\mathbb{I}_A$ denotes an identity map acting on $\mathcal{B}(\mathcal{H}_A)$. One has an obvious:

Definition 4. An entanglement witness W defined by Eq. (4) is decomposable (indecomposable, atomic, optimal, $k-EW$) if and only if the corresponding positive map Λ is decomposable (indecomposable, atomic, optimal, k positive).

It is clear that $W \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a decomposable EW if and only if

$$W = A + B^\Gamma, \tag{5}$$

where $A, B \geq 0$ and $B^\Gamma = (\mathbb{I}_A \otimes T)B$ denotes partial transposition. Witnesses which cannot be represented as in Eq. (5) are indecomposable

Let ψ be a normalize vector in $\mathcal{H}_A \otimes \mathcal{H}_B$. Denote by $R_S(\psi)$ the number of nonvanishing Schmidt coefficients of ψ. One has

$$1 \leq R_S(\psi) \leq d. \tag{6}$$

Now, W is $k-EW$ if and only if

$$\langle \psi | W | \psi \rangle \geq 0, \tag{7}$$

for each ψ such that $R_S(\psi) \leq k$. Evidently, $W \geq 0$ if and only if W is $d-EW$. Now, W is atomic if it cannot be represented as

$$W = W_1 + W_2^\Gamma, \tag{8}$$

where W_1 and W_2 are $2-EW$'s. Finally, W is optimal EW if and only if for any $P \geq 0$, $W - P$ is no longer EW. Following [6], one has the following criterion for the optimality of W: if the set of product vectors $\psi \otimes \phi \in \mathcal{H}_A \otimes \mathcal{H}_B$ satisfying

$$\langle \psi \otimes \phi | W | \psi \otimes \phi \rangle = 0 \tag{9}$$

span the total Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, then W is optimal.

C. Detecting quantum entanglement

Positive maps and EWs are basic tools in detecting quantum entanglement. A state ρ in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if for all positive maps $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$, one has

$$(\mathbb{I}_A \otimes \Lambda)\rho \geq 0. \tag{10}$$

Equivalently, if and only if for each entanglement witness W,

$$\mathrm{Tr}(\rho W) \geq 0. \tag{11}$$

Note that entangled PPT states can be detected by indecomposable maps and witnesses only. Let σ be a density operator in $\mathcal{H}_A \otimes \mathcal{H}_B$. Following [47], one introduces its Schmidt number

$$N_S(\sigma) = \min_{p_k, \psi_k}\{\max_k R_S(\psi_k)\}, \tag{12}$$

where the minimum is taken over all possible pure states decompositions

$$\sigma = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \tag{13}$$

with $p_k \geq 0$, $\Sigma_k p_k = 1$, and ψ_k are normalized vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$. Note that if $\sigma = |\psi\rangle\langle\psi|$, then $N_S(\sigma) = R_S(\psi)$. Again, one has $1 \leq N_S(\sigma) \leq d$. Suppose now that σ is PPT but entangled. Intuitively, the "weakest" quantum entangled encoded in σ corresponds to the situation when $R_S(\sigma) = R_S(\sigma^\Gamma) = 2$. Such "weakly" entangled PPT states can be detected by atomic maps and witnesses only.

III. REDUCTION MAP AS A BUILDING BLOCK

Let us start with an elementary positive map in $\mathcal{B}(\mathbb{C}^n)$ called reduction map [48]

$$R_n(X) = \mathbb{I}_n \,\mathrm{Tr}\, X - X \tag{14}$$

for $X \in \mathcal{B}(\mathbb{C}^n)$. It is well known that R_n is completely copositive (i.e., $R_n \circ T$ is CP) and optimal. Recently, this map was generalized by Breuer and Hall [36,37] to the following family of positive maps:

$$\Phi_{2k}^U(X) = \frac{1}{2(k-1)}[R_{2k}(X) - UX^T U^\dagger], \tag{15}$$

where U is an arbitrary antisymmetric unitary $2k \times 2k$ matrix. It was shown that these maps are indecomposable [36,37] and optimal [36]. Such antisymmetric unitary matrix may be easily construct as follows:

$$U = VU_0V^\dagger, \tag{16}$$

where V stands for real orthogonal matrix ($VV^\dagger = VV^T = \mathbb{I}_{2k}$) and

$$U_0 = \mathbb{I}_k \otimes J, \tag{17}$$

with J being 2×2 symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{18}$$

It is therefore clear that in this case, one has

$$\Phi_{2k}^{U}(X) = V\Phi_{2k}^{0}(V^{\dagger}XV)V^{\dagger}, \tag{19}$$

where Φ_{2k}^{0} corresponds to Φ_{2k}^{U} with $U=U_0$. Actually, one can always find a basis in $\mathbb{C}^{2k}$ such that U takes the "canonical form" U_0. Interestingly for $k=2$, the Breuer-Hall map Φ_4^0 reproduces well-known Robertson map [28] who provided it as an example of an extremal (and hence optimal [46]) indecomposable positive map. Moreover, Robertson construction may be nicely described in terms of R_2 as follows [39]:

$$\Phi_4^0\left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \frac{1}{2}\left(\begin{array}{c|c} \mathbb{I}_2 \operatorname{Tr} X_{22} & -[X_{12}+R_2(X_{21})] \\ \hline -[X_{21}+R_2(X_{12})] & \mathbb{I}_2 \operatorname{Tr} X_{11} \end{array}\right), \tag{20}$$

where $X_{kl} \in \mathcal{B}(\mathbb{C}^2)$. This pattern is reproduced for arbitrary k. It is easy to show that the action of Φ_{2k}^0 may be represented as follows:

$$\Phi_{2k}^0\left(\begin{array}{c|c|c|c} X_{11} & X_{12} & \cdots & X_{1k} \\ \hline X_{21} & X_{22} & \cdots & X_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline X_{k1} & X_{k2} & \cdots & X_{kk} \end{array}\right) = \frac{1}{2(k-1)}\left(\begin{array}{c|c|c|c} \mathbb{I}_2(\operatorname{Tr} X - \operatorname{Tr} X_{11}) & -(X_{12}+R_2(X_{21})) & \cdots & -(X_{1k}+R_2(X_{k1})) \\ \hline -(X_{21}+R_2(X_{12})) & \mathbb{I}_2(\operatorname{Tr} X - \operatorname{Tr} X_{22}) & \cdots & -(X_{2k}+R_2(X_{k2})) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline -(X_{k1}+R_2(X_{1k})) & -(X_{k2}+R_2(X_{2k})) & \cdots & \mathbb{I}_2(\operatorname{Tr} X - \operatorname{Tr} X_{kk}) \end{array}\right),$$

where again X_{ij} are 2×2 blocks. Hence, Φ_{2k}^0 is defined in Eq. (15) by R_{2k} but the above pattern shows that it basically uses reduction map R_2 only. We stress that R_2 is exceptional: it is not only optimal but also extremal. Indeed, the corresponding entanglement witness $W_2=2(\mathbb{1}\otimes R_2)P_2^+$ reads as follows:

$$W_2 = \mathbb{I}_2 \otimes \mathbb{I}_2 - P_2^+ = \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot \end{array}\right) \tag{21}$$

and $W_2=P^{\Gamma}$, where $P=|\psi\rangle\langle\psi|$ with

$$|\psi\rangle = |01\rangle - |10\rangle. \tag{22}$$

Note that R_2 may be nicely represented as follows:

$$R_2(X) = JX^TJ^{\dagger}, \tag{23}$$

with J defined in Eq. (18). Formula (23) provides Kraus representation for $R_2\circ T$ and shows that R_2 is completely copositive.

In the present paper, we propose another construction of maps in $\mathcal{B}(\mathbb{C}^{2k})$. Now, instead of treating a $2k\times 2k$ matrix X as a $k\times k$ matrix with 2×2 blocks X_{ij}, we consider alternative possibility, i.e., we consider X as a 2×2 with $k\times k$ blocks and define

$$\Psi_{2k}^0\left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \frac{1}{k}\left(\begin{array}{c|c} \mathbb{I}_k \operatorname{Tr} X_{22} & -[X_{12}+R_k(X_{21})] \\ \hline -[X_{21}+R_k(X_{12})] & \mathbb{I}_k \operatorname{Tr} X_{11} \end{array}\right). \tag{24}$$

Again, normalization factor guaranties that the map is unital, i.e., $\Psi_{2k}^0(\mathbb{I}_2\otimes\mathbb{I}_k)=\mathbb{I}_2\otimes\mathbb{I}_k$. It is clear that for $k=2$, one has

$$\Phi_4^0 = \Psi_4^0. \tag{25}$$

We stress that our new construction is much simpler than Φ_{2k}^0 and it uses as a building block the true reduction map in $\mathcal{B}(\mathbb{C}^k)$. Moreover, it is clear that it provides a natural generalization of the original Robertson map in $\mathcal{B}(\mathbb{C}^4)$.

Now, our task is to prove that Ψ_{2k}^0 defines a positive map. It is enough to show that each rank-1 projector P is mapped via Ψ_{2k}^0 into a positive element in $\mathcal{B}(\mathbb{C}^{2k})$, that is, $\Psi_{2k}^0(P)\geq 0$. Let $P=|\psi\rangle\langle\psi|$ with arbitrary ψ from $\mathbb{C}^{2k}$. Now, due to $\mathbb{C}^{2k}=\mathbb{C}^k\oplus\mathbb{C}^k$, one has

$$\psi = \psi_1 \oplus \psi_2, \tag{26}$$

with $\psi_1,\psi_2\in\mathbb{C}^k$ and hence

$$P = \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \left(\begin{array}{c|c} |\psi_1\rangle\langle\psi_1| & |\psi_1\rangle\langle\psi_2| \\ \hline |\psi_2\rangle\langle\psi_1| & |\psi_2\rangle\langle\psi_2| \end{array}\right). \tag{27}$$

One has therefore

$$\Psi_{2k}^0(P) = \frac{1}{k}\left(\begin{array}{c|c} \mathbb{I}_k \operatorname{Tr} X_{22} & -A \\ \hline -A^{\dagger} & \mathbb{I}_k \operatorname{Tr} X_{11} \end{array}\right), \tag{28}$$

where the linear operator $A:\mathbb{C}^k\to\mathbb{C}^k$ reads as follows:

$$A = |\psi_1\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_1| + \langle\psi_1|\psi_2\rangle\mathbb{I}_k. \tag{29}$$

Let $\langle\psi_j|\psi_j\rangle=a_j^2>0$ [if one of a_j vanishes then evidently one has $\Psi_{2k}^0(P)\geq 0$]. Defining

$$L = \sqrt{k}\left(\begin{array}{c|c} \mathbb{I}_k a_2^{-1} & \mathbb{O}_k \\ \hline \mathbb{O}_k & \mathbb{I}_k a_1^{-1} \end{array}\right), \tag{30}$$

one finds

$$L\Psi_{2k}^0(P)L^{\dagger} = \left(\begin{array}{c|c} \mathbb{I}_k & -\widetilde{A} \\ \hline -\widetilde{A}^{\dagger} & \mathbb{I}_k \end{array}\right), \tag{31}$$

with

$$\widetilde{A} = |\widetilde{\psi}_1\rangle\langle\widetilde{\psi}_2| - |\widetilde{\psi}_2\rangle\langle\widetilde{\psi}_1| + \langle\widetilde{\psi}_1|\widetilde{\psi}_2\rangle\mathbb{I}_k \tag{32}$$

and normalized $\widetilde{\psi}_j=\psi_j/a_j$. Hence, to show that $\Psi_{2k}^0(P)\geq 0$, one needs to prove

$$\left(\begin{array}{c|c} \mathbb{I}_k & -\tilde{A} \\ \hline -\tilde{A}^\dagger & \mathbb{I}_k \end{array}\right) \geq 0 \tag{33}$$

for arbitrary $\psi_j \neq 0$. Now, the above condition is equivalent to

$$\tilde{A}\tilde{A}^\dagger \leq \mathbb{I}_k. \tag{34}$$

Vectors $\{\psi_1,\psi_2\}$ span two-dimensional subspace in $\mathbb{C}^k$ and let $\{e_1,e_2\}$ be a two-dimensional orthonormal basis such that $\psi_1 = e_1$ and

$$\psi_2 = e^{i\lambda} s e_1 + c e_2, \tag{35}$$

with $s=\sin\alpha$ and $c=\cos\alpha$ for some angle α. Now, completing the basis $\{e_1,e_2,e_3,\ldots,e_k\}$ in $\mathbb{C}^k$, one easily finds that the matrix elements of $\tilde{A}$ has a form of the following direct sum:

$$\tilde{A} = \left(\begin{array}{c|c} e^{-i\lambda}s & c \\ \hline -c & e^{i\lambda}s \end{array}\right) \oplus e^{-i\lambda} s \mathbb{I}_{k-2}. \tag{36}$$

Hence,

$$\tilde{A}\tilde{A}^\dagger = \mathbb{I}_2 \oplus s^2 \mathbb{I}_{k-2}, \tag{37}$$

which proves Eq. (34) since all eigenvalues of $\tilde{A}\tilde{A}^\dagger$—$\{1,1,s^2,\ldots,s^2\}$—are bounded by 1.

Now, our positive maps can be useful in detecting entanglement only if they are not completely positive. It is easy to check that the corresponding Choi matrix

$$W_{2k} = (2k^2)^{(-1)} \sum_{i,j=1}^{2k} e_{ij} \otimes \Psi^0_{2k}(e_{ij}), \tag{38}$$

possesses two negative eigenvalues $\{-1/2k,(2-k)/2k^2\}$ (unless $k=2$). Hence, Eq. (38) defines true entanglement witness in $\mathbb{C}^{2k}\otimes\mathbb{C}^{2k}$. As usual using Dirac notation, we define $e_{kl} := |e_k\rangle\langle e_l|$. Note that the corresponding Brauer-Hall witness possesses only one negative eigenvalue "−1." Hence, these two classes are different (unless $k=2$).

IV. PROPERTIES OF ENTANGLEMENT WITNESSES

In this section, we study basis properties of W_{2k}.

A. W_{2k} are atomic

In order to prove that W_{2k} is atomic (and hence indecomposable), one has to define a PPT state D_{2k} such that Schmidt rank of D_{2k} and of its partial transposition D^Γ_{2k} is bounded by 2 and show that $\mathrm{Tr}(W_{2k}D_{2k})<0$. Let us introduce the following family of product vectors:

$$\phi_1 = e_1 \otimes e_1,$$

$$\phi_2 = e_1 \otimes e_{k+1},$$

$$\phi_3 = e_k \otimes e_1,$$

$$\phi_4 = e_k \otimes e_{2k}$$

$$\phi_5 = e_{k+1} \otimes e_1,$$

$$\phi_6 = e_{k+1} \otimes e_{k+1},$$

$$\phi_7 = e_{k+1} \otimes e_{2k}.$$

Define now the following positive operator:

$$D_{2k} = \frac{1}{7}(|\phi_1+\phi_6\rangle\langle\phi_1+\phi_6| + |\phi_5-\phi_4\rangle\langle\phi_5-\phi_4| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| + |\phi_7\rangle\langle\phi_7|). \tag{39}$$

One easily finds for its partial transposition

$$D^\Gamma_{2k} = \frac{1}{7}(|\phi_2+\phi_5\rangle\langle\phi_2+\phi_5| + |\phi_3-\phi_7\rangle\langle\phi_3-\phi_7| + |\phi_1\rangle\langle\phi_1| + |\phi_4\rangle\langle\phi_4| + |\phi_6\rangle\langle\phi_6|). \tag{40}$$

Now, it is clear from that both D_{2k} and D^Γ_{2k} are constructed out of rank-1 projectors and Schmidt rank of each projector is 1 or 2. Therefore,

$$N_S(D_{2k}) \leq 2, \quad N_S(D^\Gamma_{2k}) \leq 2.$$

Finally, one finds for the trace

$$\mathrm{Tr}(W_{2k}D_{2k}) = -\frac{1}{14k^2}, \tag{41}$$

which shows that W_{2k} defines atomic entanglement witness.

B. W_{2k} are optimal

To show that W_{2k} is optimal, we use the following result Lewenstein *et al.* [7]: if the family of product vectors $\psi\otimes\phi \in \mathbb{C}^{2k}\otimes\mathbb{C}^{2k}$ satisfying

$$\langle\psi\otimes\phi|W|\psi\otimes\phi\rangle = 0 \tag{42}$$

span the total Hilbert space $\mathbb{C}^{2k}\otimes\mathbb{C}^{2k}$, then W is optimal. Let us introduce the following sets of vectors:

$$f_{mn} = e_m + e_n$$

and

$$g_{mn} = e_m + i e_n,$$

for each $1 \leq m < n \leq 2k$. It is easy to check that $(2k)^2$ vectors $\psi_\alpha \otimes \psi^*_\alpha$ with ψ_α belonging to the set

$$\{e_l, f_{mn}, g_{mn}\}$$

are linearly independent and hence they do span $\mathbb{C}^{2k}\otimes\mathbb{C}^{2k}$. Direct calculation shows that

$$\langle\psi_\alpha\otimes\psi^*_\alpha|W|\psi_\alpha\otimes\psi^*_\alpha\rangle = 0, \tag{43}$$

which proves that W_{2k} is an optimal *EW*.

C. W_{2k} vs. realignment criterion

It is instructive to compare W_{2k} to other well-known PPT entanglement detection method such as, e.g., realignment (or computable cross norm) criterion [49]. It is easy to check that a family of states (39) is also detected by realignment criterion. However, it is not difficult to construct a state detected by W_{2k} but not detected by realignment. Consider the following operator in $\mathbb{C}^{2k}\otimes\mathbb{C}^{2k}$:

$$\rho_{2k} = \sum_{i,j=1}^{2k} e_{ij} \otimes \rho_{ij}^{(2k)}, \tag{44}$$

where the $2k\times 2k$ blocks are defined as follows: diagonal blocks

$$\rho_{ii}^{(2k)} = N_k\left(\begin{array}{c|c} k\mathbb{I}_k & \mathbb{O}_k \\ \hline \mathbb{O}_k & \mathbb{I}_k \end{array}\right), \tag{45}$$

for $i=1,\ldots,k$, and

$$\rho_{ii}^{(2k)} = N_k\left(\begin{array}{c|c} \mathbb{I}_k & \mathbb{O}_k \\ \hline \mathbb{O}_k & k\mathbb{I}_k \end{array}\right), \tag{46}$$

for $i=k+1,\ldots,2k$. The off-diagonal blocks are form,

$$\rho_{i,i+k}^{(2k)} = -N_k W_{i,i+k}^{(2k)}, \tag{47}$$

for $i=1,\ldots,k$,

$$\rho_{ij}^{(2k)} = N_k e_{ij}, \tag{48}$$

for $i=1,\ldots,k$, $j=k+1,\ldots,2k$, and $j\neq i+k$ and

$$\rho_{ij}^{(2k)} = \mathbb{O}_k, \tag{49}$$

otherwise. The normalization factor N_k is given by

$$1/N_k = 2k^2(k+1).$$

Direct calculation shows that

$$\rho \geq 0, \quad \rho^\Gamma \geq 0, \quad \mathrm{Tr}\,\rho = 1,$$

and one easily finds for the trace

$$\mathrm{Tr}(W_{2k}\rho_{2k}) = -\frac{k-1}{2k^3(k+1)},$$

which proves that Eq. (44) is PPT entangled. However, it is easy to check that for $k=2,3,4$, the entanglement of Eq. (44) is not detected by realignment (we conjecture that it is true for all k).

V. STRUCTURAL PHYSICAL APPROXIMATION

It is well known that positive maps cannot be directly implemented in the laboratory. The idea of *structural physical approximation* (SPA) [43,44] is to mix a positive map Λ with some completely positive map making the mixture $\tilde{\Lambda}$ completely positive. In the recent paper [45], the authors analyze SPA to a positive map $\Lambda:\mathcal{H}_A\to\mathcal{H}_B$ obtained through minimal admixing of white noise

$$\tilde{\Lambda}(\rho) = p\frac{\mathbb{I}_B}{d_B}\mathrm{Tr}(\rho) + (1-p)\Lambda(\rho). \tag{50}$$

The minimal means that the positive mixing parameter $0<p<1$ is the smallest one for which the resulting map $\tilde{\Lambda}$ is completely positive, i.e., it defines a quantum channel. Equivalently, one may introduce SPA of an entanglement witness W,

$$\tilde{W} = \frac{p}{d_A d_B}\mathbb{I}_A \otimes \mathbb{I}_B + (1-p)W, \tag{51}$$

where p is the smallest parameter for which $\tilde{W}$ is a positive operator in $\mathcal{H}_A\otimes\mathcal{H}_B$, i.e., it defines (possibly unnormalized) state.

It was conjectured [45] that SPA to optimal positive maps correspond to entanglement-breaking maps (channels). Equivalently, SPA to optimal-entanglement witnesses correspond to separable (unnormalized) states. It turns out that the family of optimal maps and witnesses constructed in this paper does support this conjecture.

The corresponding SPA of W_{2k} is given by

$$\tilde{W}_{2k} = \frac{p}{(2k)^2}\mathbb{I}_{2k} \otimes \mathbb{I}_{2k} + (1-p)W_{2k}. \tag{52}$$

Using the fact that the maximal negative eigenvalue of W_{2k} equals -1, one easily finds the following condition for the positivity of $\tilde{W}_{2k}$:

$$p \geq \frac{d}{d+1}, \tag{53}$$

with $d=2k$. Surprisingly, one obtains the same bound for p as in Eqs. (26), (33), and (65) in [45].

Now, to show that SPA of Ψ_{2k}^0 (or equivalently W_{2k}) is entanglement breaking (equivalently separable), we use the following:

Lemma 1 [50]. Let $\Lambda:\mathcal{B}(\mathbb{C}^d)\to\mathcal{B}(\mathbb{C}^d)$ be a positive unital map. Then SPA of Λ is entanglement breaking if Λ detects all entangled isotropic states in $\mathbb{C}^d\otimes\mathbb{C}^d$.

Indeed, let

$$\rho_p = \frac{p}{d^2}\mathbb{I}_d \otimes \mathbb{I}_d + (1-p)P_d^+ \tag{54}$$

be an isotropic state which is known to be entangled if and only if

$$p < \frac{d}{d+1}. \tag{55}$$

Now, assume that $(\mathbb{1}\otimes\Lambda)\rho_p$ is not positive if ρ_p is entangled. Using $\Lambda(\mathbb{I}_d)=\mathbb{I}_d$, one obtains

$$(\mathbb{1}\otimes\Lambda)\rho_p = \frac{p}{d^2}\mathbb{I}_d \otimes \mathbb{I}_d + (1-p)W, \tag{56}$$

that is, $(\mathbb{1}\otimes\Lambda)\rho_p=\tilde{W}$. Now, if $\tilde{W}$ is positive, then ρ_p has to be separable (otherwise it would be detected by Λ). But since $\mathbb{1}\otimes\Lambda$ sends separable states into separable states, one concludes that $\tilde{W}$ is separable (or equivalently $\tilde{\Lambda}$ is entanglement breaking).

Lemma 2. If in addition Λ is self-dual, i.e.,

$$\mathrm{Tr}[A\Lambda(B)] = \mathrm{Tr}[\Lambda(A)B] \tag{57}$$

for all $A,B \in \mathcal{B}(\mathbb{C}^d)$, then it is enough to check whether all entangled isotropic states are detected by the witness W, i.e., $\mathrm{Tr}(W\rho_p) < 0$ for all p satisfying Eq. (55).

Again, the proof is very easy. One has

$$\mathrm{Tr}(\rho_p W) = \mathrm{Tr}[\rho_p(\mathbb{1} \otimes \Lambda)P_d^+] = \mathrm{Tr}[(\mathbb{1} \otimes \Lambda)\rho_p P_d^+], \tag{58}$$

where in the last equality we used the self-duality of Λ. Now, if $\mathrm{Tr}(\rho_p W) < 0$ for p satisfying Eq. (55), then $(\mathbb{1} \otimes \Lambda)\rho_p$ is not positive (otherwise its trace with the projector P_d^+ would be positive). Hence by Lemma 1, SPA $\tilde{\Lambda}$ is entanglement breaking.

We are prepared to show that SPA for Ψ_{2k}^0 is entanglement breaking.

Lemma 3. Ψ_{2k}^0 is self-dual. One checks by direct calculations that

$$\mathrm{Tr}[e_{kl}\Psi_{2k}^0(e_{mn})] = \mathrm{Tr}[\Psi_{2k}^0(e_{kl})e_{mn}], \tag{59}$$

for all $k,l,m,n = 1,\ldots,2k$. Hence, due to the Lemma 2, to show that SPA for Ψ_{2k}^0 is entanglement breaking, is it enough to prove:

Lemma 4. $\mathrm{Tr}(W_{2k}\rho_p) < 0$ for all p satisfying Eq. (55).

To prove it, let us note that

$$\mathrm{Tr}(W_{2k}\rho_p) = \frac{p}{(2k)^2}\mathrm{Tr}\, W_{2k} + (1-p)\mathrm{Tr}(W_{2k}P_{2k}^+).$$

Now, $\mathrm{Tr}\, W_{2k} = 2k$ and

$$\mathrm{Tr}(W_{2k}P_{2k}^+) = \frac{1}{2k}\sum_{m,n=1}^{2k}\langle m|\Psi_{2k}^0(e_{mn})|n\rangle.$$

Finally, using definition of Ψ_{2k}^0, one gets

$$\sum_{m,n=1}^{2k}\langle m|\Psi_{2k}^0(e_{mn})|n\rangle = -2k$$

and hence

$$\mathrm{Tr}(W_{2k}\rho_p) = \frac{p(d+1)-1}{d}, \tag{60}$$

with $d = 2k$. Now, if ρ_p is entangled, i.e., $p < 1/(d+1)$, then $\mathrm{Tr}(W_{2k}\rho_p) < 0$ which shows that W_{2k} detects all entangled isotropic states.

VI. CONCLUSIONS

We have provided a new construction of *EW*s in $\mathbb{C}^d \otimes \mathbb{C}^d$ with $d = 2k$. It was shown that these *EW* are indecomposable, i.e., they are able to detect PPT entangled state. Moreover, they are so-called atomic *EW*s, i.e., they able to detect PPT entangled states ρ such that both ρ and ρ^Γ possess Schmidt number 2. The crucial property of our witnesses is their optimality, i.e., they are no other witnesses which can detect more entangled states. Interestingly, these witnesses display the so-called circulant structure studied recently in [51,52].

Equivalently, our construction gives rise to a class of positive maps in algebras of $d \times d$ complex matrices. For $k = 2$, this construction reproduces old example of Robertson map [28] and hence [39] defines the special case of Brauer-Hall maps [36,37].

Let us observe that if $\Lambda: \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$ is a positive indecomposable map, then for any unitaries $U_1, U_2: \mathbb{C}^d \to \mathbb{C}^d$, a map

$$\Lambda^{U_1U_2}(A) := U_1\Lambda(U_2^\dagger A U_2)U_1^\dagger, \tag{61}$$

is again positive and indecomposable [39]. This observation enables one to generalize Robertson map Φ_{2k}^0 and our new map Ψ_{2k}^0 to $\Phi_{2k}^{U_1U_2}$ and $\Psi_{2k}^{U_1U_2}$. Note that if $U_1 = U_2 = U$ given by Eq. (15), then $\Phi_{2k}^U := \Phi_{2k}^{UU}$ defines Breuer-Hall map. Therefore, $\Psi_{2k}^U := \Psi_{2k}^{UU}$ may be regarded as a Breuer-Hall-like generalization of our primary map Ψ_{2k}^0.

It should be stressed that an *EW* defined by the Breuer-Hall map and *EW* W_{2k} introduced in this paper are different, i.e., they do detect different classes of PPT entangled states. Direct calculation shows that the PPT entangled state (44) is not detected by the Breuer-Hall witness (for $k > 2$). On the other hand, consider the family of PPT entangled state introduced in [36]

$$\rho(\lambda) = \lambda P_d^+ + (1-\lambda)\rho_0, \tag{62}$$

with

$$\rho_0 = \frac{2}{d(d+1)}U_0 P_S U_0^\dagger, \tag{63}$$

and P_S being the projector onto the subspace of states symmetric under the swap operation. It was shown that $\rho(\lambda)$ is PPT for $0 \le \lambda \le 1(d+2)$. Moreover, Breuer-Hall witness detects all entangled states within λ family (62) [both PPT and negative partial transposition (NPT)]. Direct calculation shows that our witness W_{2k} does not detect PPT entangled states in Eq. (62).

Interestingly, the partial transposition W_{2k}^Γ defines an *EW* with $k(k-1)/2$ negative eigenvalues (all equal to -1). For $k = 2$, it gives exactly one negative eigenvalue (the fact well known from the family of Breuer-Hall maps in four dimensions). Therefore, this example provides an *EW* with multiple negative eigenvalues. However, contrary to the Breuer-Hall maps, we were not able to show that W_{2k}^Γ is optimal.

We have shown that structural physical approximation for our new class of positive maps gives rise to entanglement breaking channels. This result supports recent conjecture by Korbicz *et al.* [45].

Finally, let us mention some open questions. In this paper, we have used reduction map as a building block to construct other optimal maps. Can we use other positive maps as building blocks? Is it true that properties of building blocks (such as optimality and/or atomicity) are shared by the map which is built out of them?

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